ME751
Advanced Computational Multibody Dynamics

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Quote of the day

"Marge, It Takes Two To Lie; One To Lie and One To Listen“

-- Homer Simpson

Before we get started…

- Last Time:
  - Hopping from RF to RF
  - Describing the orientation of a body using Euler Angles
  - Connection between angular velocity and time derivatives of Euler Angles

- Today:
  - Describing the orientation of a body using Euler parameters
  - Connection between angular velocity and time derivatives of Euler parameters

- Misc. other issues
  - Check course website in case you need access to lecture slides
    - http://sbel.wisc.edu/Courses/ME751/2016/
  - Forums, registration:
    - http://sbel.wisc.edu/Forum/viewforum.php?f=19
    - https://projectchrono.org/forum/

Angular Velocity for 3-1-3 Euler Sequence

- When using the 3-1-3 Euler angles, based on the expression of **A**, one gets:

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T = (\dot{\mathbf{A}}_1\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_1\dot{\mathbf{A}}_2\mathbf{A}_3 + \mathbf{A}_1\mathbf{A}_2\dot{\mathbf{A}}_3)\mathbf{A}_3^T\mathbf{A}_2^T\mathbf{A}_1^T$$

- What you additionally know is that:

$$\left\{\begin{array}{l}
\dot{\mathbf{A}}_1 = \dot{\phi}\begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\dot{\mathbf{A}}_2 = \dot{\theta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \\
\dot{\mathbf{A}}_3 = \dot{\psi}\begin{bmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}\right.$$

Angular Velocity for 3-1-3 Euler Sequence

- If you carry out the horrific matrix multiplications you end up with the following expression for **ω**:

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\phi} + \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{bmatrix} \dot{\psi}$$

- Equivalently, you can write this in matrix form like

$$\omega = \begin{bmatrix} 0 & \cos\phi & \sin\theta\sin\phi \\ 0 & \sin\phi & -\sin\theta\cos\phi \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \equiv \mathbf{B}\dot{\epsilon}$$

…where the array of generalized coordinates ε is defined as

$$\epsilon = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

[HOMEWORK]

Useful Identities

Assume $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

- Useful identity 1 (see Eq. 9.1.29):

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\,\mathbf{I}_{3\times 3}$$

- Useful identity 2 (see Eq.9.1.30):

$$\widetilde{\tilde{\mathbf{a}}\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}}$$

Euler Parameters: Analytical Foundation

- Question – What is the theoretical underpinning of the idea of using a set of four Euler parameters to characterize the orientation attitude of a rigid body?

- Answer – The Euler Theorem:

  "If the origins of two right-hand Cartesian reference frames coincide, then the RFs may be brought into coincidence by a single rotation of a certain angle ˙ about a carefully chosen unit axis **u**"

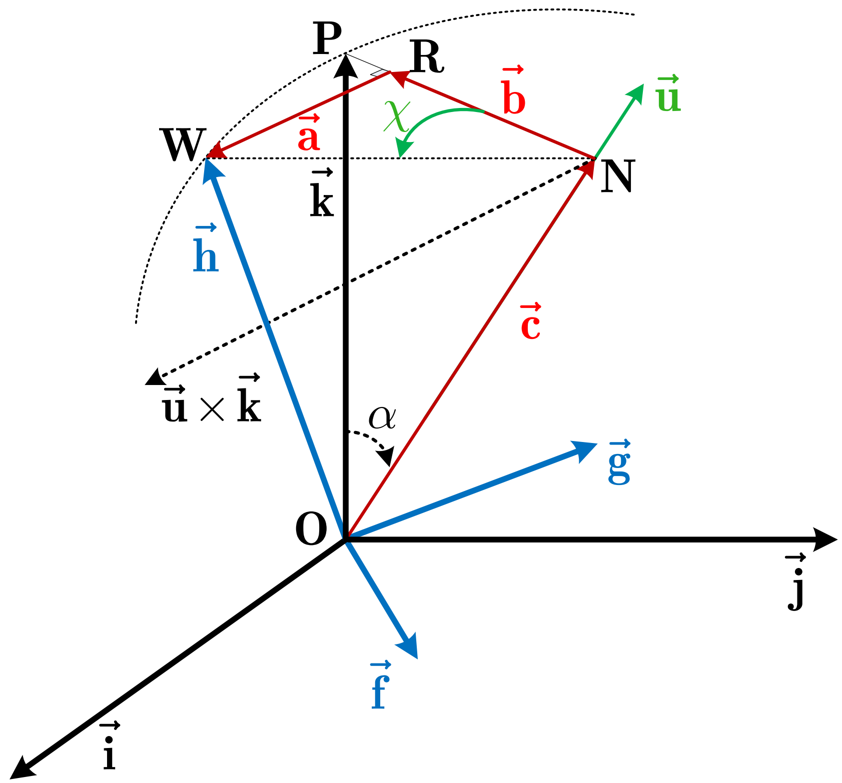
  - Euler Theorem proved in the following references:
    - Wittenburg – Dynamics of Systems of Rigid Bodies (1977)
    - Goldstein – Classical Mechanics, 2nd edition, (1980)
    - Angeles – Fundamentals of Robotic Mechanical Systems (2003)

[pp.338, Ed Haug's book]

Euler Parameters

- General remarks

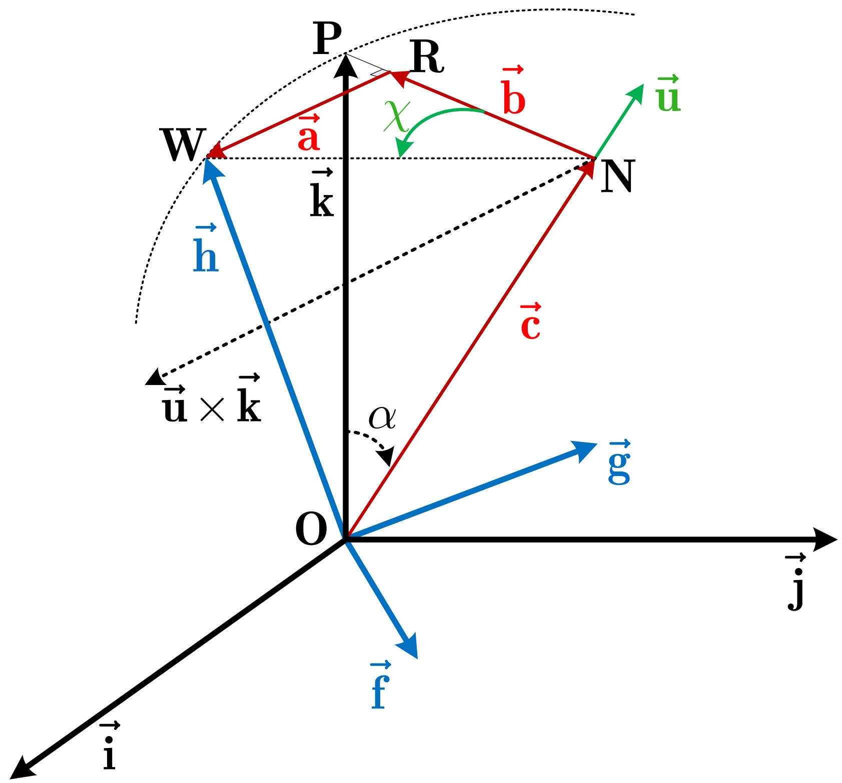
- Green color - used for quantities that define the Euler rotation: the axis of rotation defined by the **unit** vector $\vec{\mathbf{u}}$ and the angle χ
- Red color - used to indicate the vectors that need to be summed up to get axis $\vec{\mathbf{h}}$ of the L-RF
- Blue color - denotes the $\vec{\mathbf{f}} - \vec{\mathbf{g}} - \vec{\mathbf{h}}$ axes of the L-RF
- Black dotted line - support entities (helpers, don't play any role but only help with the derivation). The angle α measured between the axis of rotation $\vec{\mathbf{u}}$ and the $\vec{\mathbf{k}}$ unit vector.



- Other notation used: $||\vec{\mathbf{a}}|| = a$ $||\vec{\mathbf{b}}|| = b$ $||\vec{\mathbf{c}}|| = c$

Geometric Aspects Involved in Derivation

- $\mathbf{N}$ is the projection of $\mathbf{P}$ onto the rotation axis $\vec{\mathbf{u}}$. This is how $\vec{\mathbf{c}}$ is defined; i.e., as $\overrightarrow{\mathbf{ON}}$
- $\mathbf{R}$ is the projection of $\mathbf{W}$ onto $\mathbf{NP}$. It thus defines $\vec{\mathbf{b}}$ as $\overrightarrow{\mathbf{NR}}$, and $\vec{\mathbf{a}}$ as $\overrightarrow{\mathbf{RW}}$
- Note that $\vec{\mathbf{a}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{k}}$ are parallel since they are both perpendicular to the plane defined by the points $\mathbf{O}$, $\mathbf{N}$, and $\mathbf{P}$
- $\measuredangle ONP = \frac{\pi}{2} \quad \& \quad \measuredangle NRW = \frac{\pi}{2}$



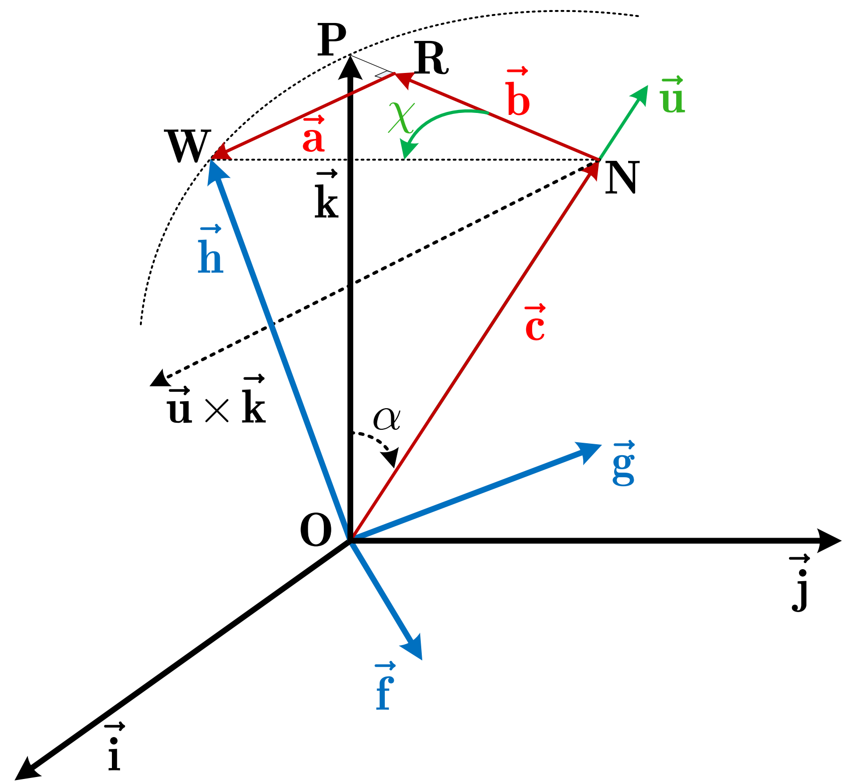
[pp.338, Ed Haug's book]

Euler Parameters

$$\measuredangle ONP = \frac{\pi}{2}$$

$$\measuredangle NRW = \frac{\pi}{2}$$

- The points **N**, **R**, **P**, **W**, as well as the vector $\vec{\mathbf{u}} \times \vec{\mathbf{k}}$ belong to the same plane
- The rotation axis $\vec{\mathbf{u}}$ is perpendicular to the plane mentioned above
- Note that $||\vec{\mathbf{u}} \times \vec{\mathbf{k}}|| = \sin\alpha$
- Note that $||NP|| = ||NW||$
- **Our Goal**: express $\vec{\mathbf{h}}$ as a function of $\vec{\mathbf{k}}$ and the parameters associated with the rotation; i.e., $\vec{\mathbf{u}}$ and χ



- The starting point:

$$\vec{\mathbf{h}} = \vec{\mathbf{c}} + \vec{\mathbf{b}} + \vec{\mathbf{a}}$$

$$\mathbf{a} = \tilde{\mathbf{u}}\mathbf{k}\,\sin\chi$$

$$\mathbf{b} = [\mathbf{k} - (\mathbf{u}^T\mathbf{k})\mathbf{u}]\cos\chi$$

$$\mathbf{c} = (\mathbf{u}^T\mathbf{k})\mathbf{u}$$

$\Downarrow$

$$\mathbf{h} = \mathbf{k}\cos\chi + (\mathbf{u}^T\mathbf{k})\mathbf{u}(1-\cos\chi) + (\tilde{\mathbf{u}}\mathbf{k})\sin\chi$$

$\Downarrow$

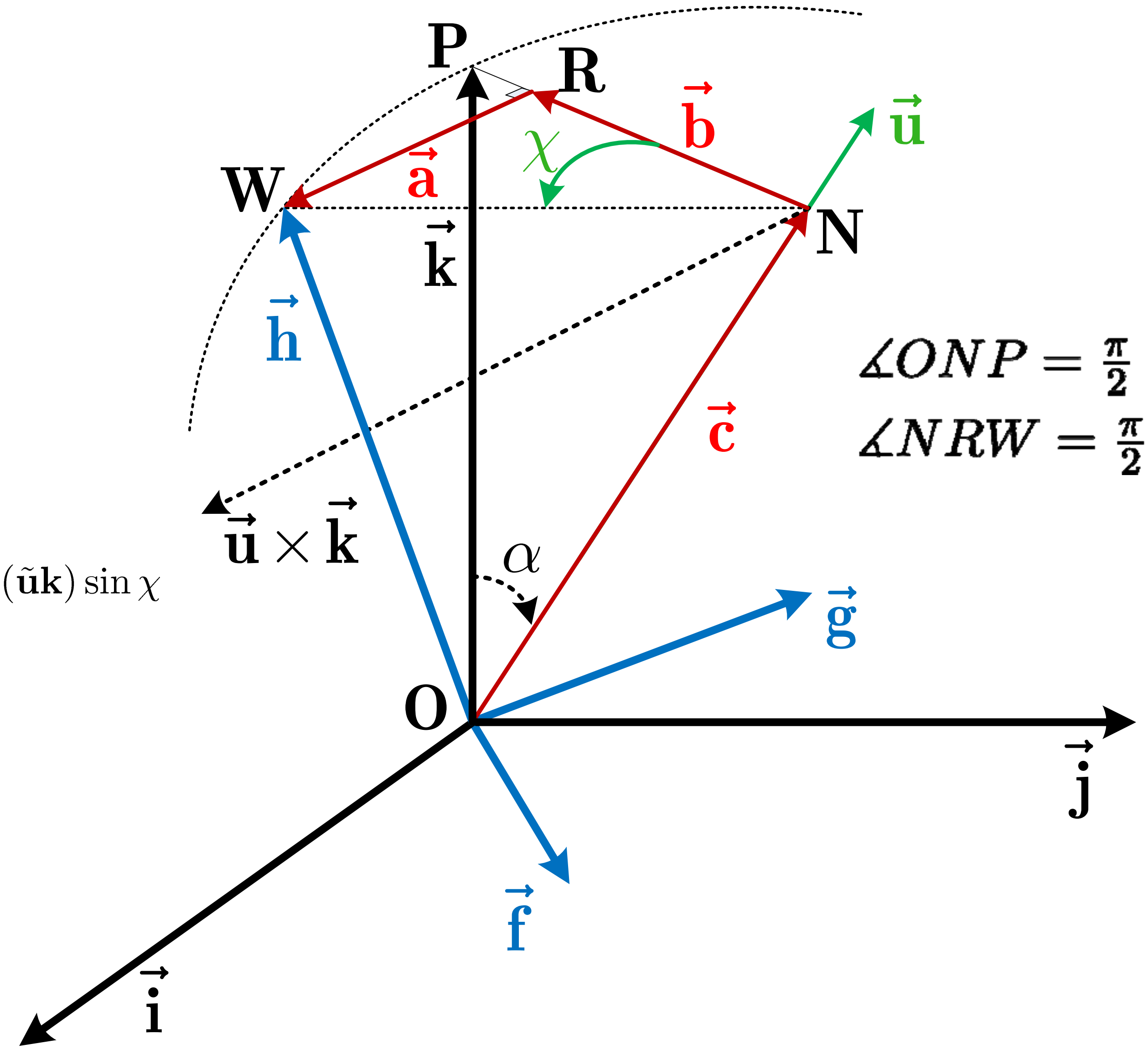
$$1 - \cos\chi = 2\sin^2\tfrac{\chi}{2}$$

$$\sin\chi = 2\sin\tfrac{\chi}{2}\cos\tfrac{\chi}{2}$$

$$\cos\chi = 2\cos^2\tfrac{\chi}{2} - 1$$

$\Downarrow$

$$\mathbf{h} = \mathbf{k}(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{k})\sin^2\frac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{k}\sin\frac{\chi}{2}\cos\frac{\chi}{2}$$



Euler Parameters: Definition

- Note that following *exactly* the same steps for axes **f** and **g** we eventually get:

$$\mathbf{f} = \mathbf{i}(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{i})\sin^2\frac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{i}\sin\frac{\chi}{2}\cos\frac{\chi}{2}$$

$$\mathbf{g} = \mathbf{j}(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{j})\sin^2\frac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{j}\sin\frac{\chi}{2}\cos\frac{\chi}{2}$$

$$\mathbf{h} = \mathbf{k}(2\cos^2\frac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{k})\sin^2\frac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{k}\sin\frac{\chi}{2}\cos\frac{\chi}{2}$$

- Recall that we want to express everything in terms of **u** $=[u_1, u_2, u_3]^T$ and ·
- The entries in the expressions of **f**, **g**, and **h** motivate the decision to introduce the following generalized coordinates (the "Euler Parameters"):

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad \text{where} \qquad e_0 = \cos\frac{\chi}{2} \quad \text{and} \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \mathbf{u}\sin\frac{\chi}{2}$$

Euler Parameters: Quick Remarks

- Recall that **u** is a unit vector. Therefore, the following holds

$$\mathbf{p}^T\mathbf{p} = e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \qquad (1)$$

  - Identity above referred to as the "Euler Parameter normalization constraint"

- The Euler Parameters e_1, e_2, and e_3 are typically grouped together and denoted by **e**:

$$\mathbf{e} \equiv \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \mathbf{u}\sin\frac{\chi}{2}$$

- Then, Eq.(1) becomes

$$\mathbf{p}^T\mathbf{p} = e_0^2 + \mathbf{e}^T\mathbf{e} = 1$$

Using Euler Parameters to Express Orientation Matrix A

- Recall that **A**=[**f g h**]
- Based on the definition of e_0, e_1, e_2, and e_3, one has that

$$\begin{aligned}
\mathbf{f} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{i} \\
\mathbf{g} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{j} \\
\mathbf{h} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{k}
\end{aligned}$$

$$\Downarrow$$

$$\boxed{\mathbf{A} = [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]} \qquad (1)$$

- Equivalently, Eq.(1) can be explicitly expressed using e_0, e_1, e_2, e_3 as

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

[one slide detour]

Position Level Identities

[Level 0 Identities][p.343]

- There are two 3×4 matrices that show up quite often:

$$\mathbf{E} \equiv [-\mathbf{e}, \tilde{\mathbf{e}} + e_0 \mathbf{I}] = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \qquad \mathbf{G} \equiv [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0 \mathbf{I}] = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}$$

- The key identity is this [HOMEWORK]:

$$\mathbf{A} = \mathbf{E}\mathbf{G}^T$$

- Also note the following five identities:

1. $\mathbf{E}\mathbf{p} = \mathbf{0}_3$
2. $\mathbf{G}\mathbf{p} = \mathbf{0}_3$
3. $\mathbf{E}\mathbf{E}^T = \mathbf{G}\mathbf{G}^T = \mathbf{I}_3$
4. $\mathbf{G}^T\mathbf{G} = \mathbf{E}^T\mathbf{E} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^T$
5. $\tilde{\mathbf{e}}\tilde{\mathbf{e}} = \mathbf{e}\mathbf{e}^T + (e_0^2 - 1)\,\mathbf{I}_{3\times 3}$ (see today's "Useful identity 1")

Euler Parameters: Putting Things In Perspective

- I can count on a rotational angle ˈ and an axis of rotation **u** to take the G-RF into an L-RF (Euler's Theorem)

- I used ˈ and **u** to define a set of four generalized coordinates
  - Called them Euler Parameters **p**

- I managed to express the rotation matrix **A** in terms of my choice of generalized coordinates

- I have to keep in mind that there is a constraint normalization condition that **p** must satisfy (its norm is one)

- It remains to prove that there is a one-to-one mapping between **A**; i.e., the orientation of the body, and our choice **p** of GCs

Euler Parameters: The One-To-One Mapping to A

- One-To-One Mapping: What are we interested in?

  - PART 1: We'd like to see that given a set of "healthy" Euler Parameters **p**, they lead to a unique orientation matrix **A**
    - Healthy **p** : it satisfies the constraint $\mathbf{p}^{T} ƒ \mathbf{p} = 1$
    - Healthy **A**: the matrix is orthonormal ($\mathbf{A}^{T} ƒ \mathbf{A} = \mathbf{I}_3$)

  - PART 2: Conversely, we'd like to see that given a rotation matrix **A**, that is, an orientation of the rigid body, there is a unique set of "healthy" Euler Parameters **p** that are associated with that orientation of the body
    - Recall that the 3-1-3 Euler Angle representation didn't cut it when „=k `

Part 1: $\mathbf{p} \Rightarrow \mathbf{A}$

The p and A One-To-One Mapping

- Given a set of "healthy" Euler Parameters **p**, we should prove that they lead to a unique and healthy orientation matrix **A**
  - Healthy **p** : it satisfies the constraint $\mathbf{p}^T \mathbf{p} = 1$
  - Healthy **A**: the matrix is orthonormal ($\mathbf{A}^T \mathbf{A} = \mathbf{I}_3$)

- Recall that

$$\mathbf{A} = [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})] = 2 \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix} \quad (1)$$

- Since $e_0^2 + \mathbf{e}^T\mathbf{e} = 1$, it follows that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$:

- Moreover, according to Eq. (1), it's obvious that given $p=[e_0, e_1, e_2, e_3]^T$ the matrix **A** is unique

Part 2: $\mathbf{A} \Rightarrow \mathbf{p}$
The p and A One-To-One Mapping

- Need to prove that given a rotation matrix **A**, that is, an orientation of the rigid body, there is a unique set of “healthy” Euler Parameters **p** that are associated with that orientation of the body

- The basic idea of the analysis: given the entries a_{11}, …, a_{33} of a healthy **A**, we should show that they lead to a unique **p**.
  - If it turns out that **p** is not unique, can we live with it?

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 2 \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

These nine entries are given

Can you get e_0 , e_1 , e_2 , e_3 uniquely?

[pp.341]

Part 2: $\mathbf{A} \Rightarrow \mathbf{p}$
Two Possible Cases

- Note that

$$\left.\begin{aligned} \text{trace}(\mathbf{A}) &= a_{11} + a_{22} + a_{33} \\ \text{trace}(\mathbf{A}) &= 2(3e_0^2 + e_1^2 + e_2^2 + e_3^2) - 3 = 4e_0^2 - 1 \end{aligned}\right\} \Rightarrow e_0^2 = \frac{\text{trace}(\mathbf{A}) + 1}{4}$$

- Based on the value of e_0 we have two cases associated with Part 2:
  - Case 1: $e_0 \neq 0$
    - Note that in this case the sign of e_0 is arbitrary and not important (more later)
    - Nonetheless, in what follows we'll have to stick with that sign decision
  - Case 2: $e_0=0$

[pp.341]

Part 2, Case 1: $e_0 \neq 0$

- Recall that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 2 \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

- Then, with $e_0 \neq 0$ available to you (see previous slide), you get

$$a_{32} - a_{23} = 4e_0 e_1$$
$$a_{13} - a_{31} = 4e_0 e_2$$
$$a_{21} - a_{12} = 4e_0 e_3$$

- From where,

$$e_1 = \frac{a_{32} - a_{23}}{4e_0}$$
$$e_2 = \frac{a_{13} - a_{31}}{4e_0}$$
$$e_3 = \frac{a_{21} - a_{12}}{4e_0}$$

“p” or “–p”?

- Question: Say you choose $e_0^+ = a > 0$ and get $\mathbf{p}^+$ as outlined before. What happens if you chose $e_0^- = -a < 0$? How would $\mathbf{p}^-$ look like?
- Answer: it turns out that $\mathbf{p}^- = -\mathbf{p}^+$
  - This is immediate to see, since

$$
\begin{aligned}
e_1^- &= \tfrac{a_{32} - a_{23}}{-4a} \\
e_2^- &= \tfrac{a_{13} - a_{31}}{-4a} \\
e_3^- &= \tfrac{a_{21} - a_{12}}{-4a}
\end{aligned}
$$

- There is yet a much more interesting meaning to the decision of choosing $e_0 = -a < 0$. The resulting Eule Parameter associated with $\mathbf{p}^-$ describes a rotation of angle $2\pi - \chi$ around the axis $-\vec{\mathbf{u}}$.
  - Note that this rotation brings the body to *exactly* the same attitude

[AO]
Exercise

- Assume that a rigid body rotation of angle χ and unit axis $\vec{\mathbf{u}}$ leads to the set of Euler Parameters $\mathbf{p}$.

  1. Prove that the set of Euler Parameters $-\mathbf{p}$ correspond to a rotation of angle $2\pi - \chi$ around the axis $-\vec{\mathbf{u}}$
  2. Prove that the set of Euler Parameters $-\mathbf{p}$ leads in fact to the same orientation matrix $\mathbf{A}$

Putting Things in Perspective…

- Recall that so far we considered the case $e_0 \neq 0$ only. What have we found?
  - We found a way to compute $\mathbf{p}$ given $\mathbf{A}$.
  - The $\mathbf{p}$ that we found satisifies the Euler Parameter normalization constraint
  - We have a little bit of a problem, in that we actually found two sets: $\mathbf{p}^+$ and $\mathbf{p}^-$, that originate from the same $\mathbf{A}$; i.e., from the same orientation of the rigid body.
    * However, we can live with this, since the two are quite different and there is no danger of confusing one for the other (we'll increment values of $\mathbf{p}$ by small values when we do kinematics/dynamics analysis and a continuity argument will tell us which $\mathbf{p}$ we have to work with)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 2 \begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

[pp.341]

Part 2, Case 2: $e_0 = 0$

- Since $e_0 = \cos\frac{\chi}{2} = 0$, it means that $\chi = \pi$.
  - Note that we assume that χ is in between 0 and 2π
- How do you compute e_1, e_2, e_3 next?
- Their computation draws on the following remark: since now $e_1^2 + e_2^2 + e_3^2 = 1$, at least one of e_1, e_2, and e_3 is nonzero. Whichever that one is, you'll use it to solve for the other two using two out of the three following conditions:

$$a_{21} + a_{12} = 4e_1e_2$$

$$a_{31} + a_{13} = 4e_1e_3$$

$$a_{32} + a_{23} = 4e_2e_3$$

  - Note that you'll have the same sign issue like we saw for the $e_0 \neq 0$ case. However, we understand what that means and what its implications are

Euler Parameters: The One-To-One Mapping to A – Concluding Remarks

- One-To-One Mapping, Euler Parameter to Body Orientation, final word:

  - Things worked out well…

…(with the caveat that you have for each **A** two **p** sets, yet they are far apart and there is no danger to get confused)

[Challenge Homework]
Exercise

- Assume that given $\mathbf{A}$, you have just obtained e_0, e_1, e_2, e_3, as indicated before. For this exercise, we'll assume that indeed $e_0 \neq 0$. Prove that these values satisfy the Euler Parameter normalization constraint. That is,

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

- Hint: Use the fact that the matrix $\mathbf{A}$ is orthonormal and its determinant is 1.
- Even more interesting: prove that for the $\mathbf{p}$ you just got, if you do $\mathbf{E}\mathbf{G}^T$ you get back $\mathbf{A}$
- Keep in mind that you have two cases: first, when $e_0 \neq 0$; second, when $e_0 = 0$
- NOTE: I couldn't find a proof for this anywhere I looked

[New Topic]
Euler Parameter Identities

- We'll discuss a couple of identities (formulas) that involve the Euler parameters. Why?
  - They are needed for the discussion regarding the angular velocity
  - They are needed when discussing about the concept of virtual displacement
- There is very little intuition behind these identities (at least to me)
  - Take them as they are: some helpers who are going to show up here and there in proving various results that involve the matrix **A** or its time derivatives